Macro-orbitals and Microscopic Theory of a System of Interacting Bosons

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Abstract

Macro-orbital representation of a particle in a many body system is used to develop the microscopic theory of a system of interacting bosons. Each particle in the system represents a pair of particles moving with equal and opposite momenta $(\mathbf{q}, -\mathbf{q})$ at their center of mass (CM) which moves with momentum $\mathbf{K}$ in the laboratory frame. Below certain temperature (say, T_λ), these particles assume a state of $(\mathbf{q}, -\mathbf{q})$ bound pairs (named as SMW pairs). The $\lambda-$transition is found to be a consequence of inter-particle quantum correlations clubbed with zero-point repulsion and inter-particle attraction; it is an onset of the *order-disorder* of particles in their ϕ-space followed simultaneously by their *Bose Einstein condensation* as SMW pairs in a state of $q = q_o = \pi/d$ and $K = 0$. Particles at $T \leq T_\lambda$ acquire collective binding which locks them at $< k >= 0$ in momentum space, $< r >= \lambda/2$ in real space and at $\Delta\phi = 2n\pi$ (with $n = 1, 2, 3, ...$) in $\phi-$space. Consequently, the entire system assumes *mechanical strain* in inter-particle bonds and behaves like a single macroscopic molecule. The collective binding is identified as an energy gap between the superfluid and normal fluid phases. The fractional density of condensed particles ($n_{K=0}(T)$) varies smoothly from $n_{K=0}(T_\lambda) = 0$ to $n_{K=0}(0) = 1.0$. The $\lambda-$transition represents the occurrence of twin phenomena of broken gauge symmetry and phase coherence. In variance with the conventional belief, the system does not have single particle $p = 0$ condensate. In addition to the well known modes of collective motions (such as phonons), the superfluid also exhibits a new kind of quantum quasi-particle, *omon*, (a phononlike wave of the oscillations of momentum coordinates of the particles. Omons are sustained because the strain energy of inter-particle bonds is a function of relative momentum of particles. The theory explains the properties of $He - II$, including the origin of quantized vortices, critical velocities, logarithmic singularity of specific heat and related properties at quantitative level. It conforms to the excluded volume condition, microscopic and macroscopic uncertainty, and renders microscopic foundation to two fluid theory of Landau, Ψ-theory of Ginzburg, the idea of macroscopic wave function of London, *etc*. The framework of this theory also helps in unifying the physics of widely different systems of interacting bosons and fermions.

1.0 INTRODUCTION

Liquid ^4He (LHE-4), - a typical *system of interacting bosons* (SIB), has been investigated extensively for its unique properties at *low temperature* (LT) and results are reviewed in several articles (*e.g.* [1-10] and other references cited therein). It transforms from its normal (N) phase ($He - I$) to superfluid (S) phase ($He - II$) at $T_\lambda = 2.17$ K with the latter exhibiting several unique properties including superfluidity. One finds that these properties can not be explained as the properties of a classical liquid and the same is true for the fact that helium remains liquid down to the lowest attainable *temperature* (T). Consequently, these properties are, rightly, related to the manifestation of wave nature of particles at macroscopic level and LHE-4 provides unique opportunity to study this aspect. Naturally, different studies of a SIB are considered to be of fundamental importance.

Soon after the discovery of superfluidity of $He-II$ [11], London [12] proposed that the phenomenon arises from $p = 0$ *condensate*, -a macroscopically large fraction ($n_{p=0}(T)$) of ^4He atoms occupying a single particle state of momentum $p = 0$. To this effect, London was guided by the well known conclusion that a *system of non-interacting bosons* (SNIB) should exhibit a transition identified as *Bose Einstein condensation* (BEC) at certain temperature (T_{BEC}) [5, 13] and the LT phase should have non-zero $n_{p=0}(T)$, -increasing smoothly from $n_{p=0}(T_{BEC}) = 0$ to $n_{p=0}(T = 0) = 1.0$ through $n_{p=0}(T < T_\lambda) < 1.0$. The idea was initially criticized by Landau [14] because LHE-4 is not a SNIB. But the Bogoliubov's theory of weakly interacting bosons [15], concluding that the increasing strength of interactions *simply* depletes $n_{p=0}(T)$, provided some ground. Consequently, different mathematical models such as Bogoliubov prescription [15], pseudo-potential technique [16], Jastrow formalism [17, 18], Feenberg's perturbation [2, 18], *etc.* have been used to find the magnitude of $n_{p=0}(T)$ in $He - II$ and at the same time develop a general framework of microscopic theories hereafter known as *conventional theories* (CTs). While several articles published in [10] discuss the role of BEC to understand the LT behavior of widely different many body quantum systems, papers by Moroni *et. al.* [19] and Sokol [20] make useful conclusions in relation to our conventional understanding of LHE-4. It appears that CTs follow two different approaches, **A1** and **A2**.

A1 : This approach developed by Bogoliubov [15], Beliaev [21, 22], Hugenholtz and Pines [23], Lee and Yang [24], Brueckner and Sawada [25], Wu [26], Sawada [27], Gavoret and Nozieres [28], and Hohenberg and Martin [29], DeDominicis and Martin [30] has been reviewed by Woods and Cowley [4] and Toyoda [31]. While it is elegantly introduced by Fetter and Wadecka [32], its important aspects are summed up, recently, by Nozieres [33], and Huang [34]. A theoretical formulation following this approach starts with the hamiltonian written in terms of second quantized Schrödinger fields and proceeds by using important inferences such as: (i) the *hard core* (HC) potential can be used perturbatively by using a method such as pseudopotential method [16], (ii) coupling constant can be

related directly to the two body *scattering length* (σ) [22], (iii) a dimensionless expansion parameter ($n\sigma^3$) can be used to make perturbative calculation [31], and (iv) the "depletion" of $n_{p=0}(T)$ can be treated in the conventional perturbation by introducing c-number operator for $p = 0$ bosons [35]. Using the quantum statistical mechanical expectation value of the second quantized field operator as an *order parameter* (OP) of the transition [23, 29], it then calculates the relevant part of *free energy* as a function of this OP and explains superfluidity and related properties. To this effect, one uses sophisticated mathematical tools as discussed in [31, 36].

A2 : This approach aims at finding the *radial distribution function* [$g(r)$] and liquid structure factor [$S(Q)$] which can be used to calculate the *ground state* (G-state) properties, excitation spectrum, different thermodynamic properties and equation of state. Superfluidity and related properties are explained in terms of one body density matrix $\rho(r)$ whose asymptotic value at large r gives the condensate fraction $n_{p=0}(T)$ [37]. While different aspects of this approach and related subject are discussed elegantly by Feenberg [2], Croxton [38], Ceperley and Kalos [39] and Yang [37] and its application to LHE-4 has been reviewed by Smith *et. al.* [40(a)], Campbell and Pinski [40(b)], Schmidt and Pandharipande [40(c)], Reatto [40(d)], and Ristig and Lam [40(e)], several methods of computer calculation of $g(r)$, $S(Q)$ and related properties are discussed and reviewed in [41-43]. The approach has been used by Moroni *et. al.* [19] and Kallio and Piilo [44] to determine the properties of LHE-4 and electron gas, respectively. While a comprehensive list of important papers related to this approach are also available in [19, 44], developments in our conventional understanding of quantum fluids can be seen in most recent books and reviews [45-48].

Over the last seven decades a large number of papers have been published [49] to establish the existence of $n_{p=0}(T)$ in $He - II$ and to find its value. However, several prominent scientists working in the field have expressed their doubts about the success of these approaches in concluding the desired theory. For example, reviewing the progress of theoretical work at the fifteenth Scottish university summer school (1974), Rickayzen [50] states, "There is no microscopic theory of superfluid ^4He. There are many mathematical models which appear to provide insight into the behavior of superfluids but there is no theory which provides quantitative prediction that agree with observation. Even some of the most widely held assumptions of the theory such as the idea of condensation in a zero momentum state can not be said to be proved beyond reasonable doubt." In their review on the subject, Woods and Cowley [4] also observe, "Despite all the experimental information and the numerous theoretical discussions there is still no convincing theory of the excitations which begins with the known interaction between helium atoms." Interestingly, though Kleban [51] proved that theories assuming the existence of $p = 0$ condensate contradict the *excluded volume condition* (EVC),- a direct consequence of HC nature of particles but his inference did not receive due importance. Similarly, Sokol [20] makes interesting observations which doubt the accuracy of the inference about the existence of

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$p = 0$ condensate in $He - II$. He observes that: (i) neutron inelastic scattering experiments have so far not given any indication of the direct evidence (*i.e.* a δ-function peak in the momentum distribution (n_p) at $p = 0$) of the existence of $n_{p=0}(T)$ in $He - II$ and it is unlikely that this goal will ever be reached, and (ii) in the absence of detailed microscopic theory the estimate of $n_{p=0}(0) \approx 0.1$ from different experiments depends on current theories, models, and empirical relations used to interpret experimental results and if their underlying assumptions are incorrect, such estimates of $n_{p=0}(T)$ will have no meaning. In addition though significant progress has been made in refining our mathematical methods and computational techniques applied to many body systems (including liquid ^4He) as one finds from most recent books and reviews [45-48] but the presumed existence of $p = 0$ condensate remains central to our present understanding of the superfluidity of $He - II$ and similar SIB(s); this is also evident from the most recent books [52] summarizing the important aspects of LHE-4.

As such one finds that the best possible model that could be used to understand superfluidity is the $\Psi-$theory [53] (a union of the two fluid model of Landau [14], quantized circulation [54], vanishing of superfluid density at the boundaries of the system [5], *etc.*) which has emerged as exceedingly successful phenomenology. However, the microscopic theory of a phenomenon has greater significance because only its agreement with experiments testifies the validity of the basic theory of nature, the above mentioned observations motivated us to look for an alternative approach. To this effect our initial reports [55-58] projecting the basic aspects of our approach greatly helped us in crystallizing our scientific ideas, refining our physical arguments and mathematical formulations and laying down the basic foundations for the final form of our theory that we report in this paper.

Our theory is based on the macro-orbital representation of a particle which we concluded in our recent paper [59] related to the wave mechanics of two HC particles in 1-D box and used in our studies related to the unification of the physics of fermionic and bosonic systems [60], ground stated of N HC core particles in 1-D box [61] and basic foundations of the microscopic theory of superconductivity [62]. Our theory makes no assumption about the basic factors (*viz.* the existence of $p = 0$-condensate or bound pair formation) which could, presumably, be responsible for the superfluidity of a SIB. In stead we only use the solutions of our $N-$body Schrödinger equation to reach our conclusions. In fact our theory, for the first time, use the results of an obvious wave mechanical superposition of two particles to conclude the orderly placement of particles in phase space of the system and this simple effort helped us in finding an almost *exact theory* of a SIB which explains all unique properties of $He - II$. In variance with CTs, our theory concludes that the origin of superfluidity and related properties of a SIB lies with the BEC condensation of $(\mathbf{q}, -\mathbf{q})$ bound pairs in a state of their center of mass (CM) momentum $\mathbf{K} = 0$.

The paper has been arranged as follows. Analyzing the N-particle microscopic quantum hamiltonian, Section (2.0) identifies a pair of particles as the basic unit of the system,

uses macro-orbital representation (Appendix -A) of a particle in writing $N-$particle state function and concludes G-state configurations of a SIB and related aspects. While Section (3.0) analyzes the thermodynamic behavior of the system, Section (4.0) reports two different approaches used to study thermal excitations. While the origin of (**q**, -**q**) bound pair formation, energy gap, *etc.*, are studied in Section (5.0), the consequences of the gap are examined in Section (6.0) and the consistency of our theory with: (i) phenomenological theories and (ii) experimental results on LHE-4 is analyzed in Section 7.0. Finally, we present some important concluding remarks in Section 8.0

2.0 Basic Foundations of Our Theory

2.1 Hamiltonian

A system of N interacting bosons such as liquid ^4He can be described, to a good approximation, by

$$H(N) = \sum_i^N h_i + \sum_{i<j} V(r_{ij}) \qquad \text{with} \quad h_i = -(\hbar^2/2m)\bigtriangledown_i^2, \qquad (1)$$

where m is the mass of a particle and $V(r_{ij})$ is a two body *central force potential* which can be expressed as the sum of: (i) a short range strong repulsion $V^R(r_{ij})$ and (ii) a weak attraction $V^A(r_{ij})$ of slightly longer range. To a good approximation, while $V^R(r_{ij})$ can be equated to *hard core* (HC) interaction $V_{HC}(r_{ij})$ [defined by $V_{HC}(r_{ij} < \sigma) = \infty$ and $V_{HC}(r_{ij} \geq \sigma) = 0$ with σ being the HC diameter of a particle], $V^A(r_{ij})$ can be replaced by a constant negative potential, *say* $-V_o$. Although, this seems to imply that the main role of $V^A(r_{ij})$ is to keep particles within volume (V) of the system but as concluded in Section (5.0) it also plays an important role in binding two HC bosons having equal and opposite (**q**, -**q**) momenta. However, to begin with, our system can be identified as an ensemble of N HC bosons confined to a volume V and its hamiltonion can be written as,

$$H_o(N) = \sum_i^N h_i + \sum_{i>j}^N A\delta(r_{ij}). \qquad (2)$$

where we use

$$V_{HC}(r_{ij}) \equiv A\delta(r_{ij}), \qquad (3)$$

with A being the strength of Dirac delta repulsion. For impenetrable HC particles, we have $A \to \infty$ when $r_{ij} \to 0$. While the equivalence expressed by Eqn.3 is mathematically shown by Huang [63], its physical basis can be understood by examining the possible configuration of two HC particle (*say*, P1 and P2) right at the instant of their collision. When P1 and P2 during a collision have their individual CM located, respectively, at $r_{CM}(1) = \sigma/2$ and $r_{CM}(2) = -\sigma/2$ (with r_{CM} being the distance of the CM of a particle from the CM of the pair of P1 and P2), they register their physical touch at $r = 0$ and their encounter with $V_{HC}(r_{ij})$ is a result of this touch beyond which the two can not be

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pushed in. The process of collision only identifies this touch; it does not register how far are the CM points of individual particles at this instant. In other words the rise and fall of the potential energy of P1 and P2 during their collision at $r = 0$ is independent of their σ and this justifies $V_{HC}(r_{ij}) \equiv A\delta(r_{ij})$. It may, however, be mentioned that this equivalence will not be valid in accounting for certain physical aspects of the system (*e.g.*, the volume occupied by a given number of particles) where the real size of the particle assumes importance.

2.2 Basic unit of the system

In what follows from Section 2.1, the motion of each particle in the system can be described by a plane wave

$$u_{\mathbf{p}}(\mathbf{b}) = \mathrm{A}\exp(i\mathbf{p}.\mathbf{b}) \tag{4}$$

unless it collides with other particle(s). Here A is normalization constant, while $\mathbf{p}$ and $\mathbf{b}$, respectively, are momentum (*wave number*) and position vectors of the particle. However, the plane wave description is modified when the particle collides with its neighbors or boundary walls of the system. A collision of a particle could either be a two body collision or a many body collision (*e.g.*, two mutually colliding particles also collide simultaneously with other particle(s)). In the former case two colliding particles (P1 and P2) simply exchange their momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ or positions $\mathbf{b}_1$ and $\mathbf{b}_2$ without any difference in the sum of their pre- and post-collision energies. However, in the latter case P1 and P2 could be seen to jump from their state of $\mathbf{p}_1$ and $\mathbf{p}_2$ to that of different momenta $\mathbf{p}'_1$ and $\mathbf{p}'_2$ (possibly of different energy) but it is clear that to a good approximation they remain in one of the possible states of two HC particles moving in the absence of other particle(s). Evidently, the complex dynamics of the system can be described, to a good approximation, in terms of the simple dynamics of a pair of HC particles as its *basic unit* and this is consistent with the fact that inter-particle forces in our system are basically two body forces. We analyze the wave mechanics of a pair of HC particles in Appendix-A and use its inferences in the following analysis.

2.3 State function

As concluded in Appendix-A, each particle in our system is represented more accurately by a macro-orbital (*cf.* Eqn. A-17) rather than by $u_{\mathbf{p}}(\mathbf{b})$. Accordingly, each particle is a member of the pair of particles moving with equal and opposite momenta ($\mathbf{q}$, - $\mathbf{q}$) at their CM which moves with moment $\mathbf{K}$ in the laboratory frame. It has two motions, q and K. While q defines its quantum size ($\lambda/2 = \pi/q$) representing the size of the real space occupied exclusively by it, K defines its motion as a free particle of mass $4m$. By using N macro-orbitals for N particles, we obtain

$$\Psi_n^j(N) = \Pi_i^N \zeta_{q_i}(r_i) \sum_P^{N!} [(\pm 1)^P \Pi_i^N \exp i(P\mathbf{K}_i.\mathbf{R}_i)] \tag{5}$$

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as the state function which represents one of the $N!$ microstates of the system. Here we have $\zeta_{q_i}(r_i) = \sin(\mathbf{q}_i.\mathbf{r}_i)$ and $\sum_P^{N!}$ referring to the sum of $N!$ product terms obtainable by permuting N particles on different $\mathbf{K}_i$ states with $(+1)^P$ and $(-1)^P$, respectively, used for selecting a symmetric and anti-symmetric wave function for an exchange of two particles. In principle, the permutation of N particles on different $\mathbf{q}_i$ states also gives $N!$ different $\Psi_n^j(N)$ and this renders

$$\Phi_n(N) = \frac{1}{\sqrt{N!}} \sum_j^{N!} \Psi_n^j(N) \tag{6}$$

as the complete wave function of a possible quantum state of the system.

2.4 State energy

Since each macro-orbital representing a particle (say $i-$th) in $\Phi_n(N)$ state has a reference to the CM coordinate system related to the pair of which the $i-$th particle is a member, we need to recast $H_o(N)$ (Eqn. 2) in terms of these coordinates in order to examine if $\Phi_n(N)$ is an eigenfunction of $H_o(N)$. To this effect, we define

$$h(i) = \frac{1}{2}[h_i + h_{i+1}] = -\frac{\hbar^2}{8m} \nabla_{R_i}^2 - \frac{\hbar^2}{2m} \nabla_{r_i}^2 \tag{7}$$

with $h_{N+1} = h_1$ and write Eqn. 2 as

$$H_o(N) = \sum_i^N h(i) + \sum_{i>j}^N A\delta(r_{ij}). \tag{8}$$

This easily renders [58-62]

$$< \Phi_n(N)| \sum_{i>j}^N A\delta(r_{ij})|\Phi_n(N) >= 0 \tag{9}$$

and

$$E_n =< \Phi_n(N)|H_o(N)|\Phi_n(N) >= \sum_i^N \left[\frac{\hbar^2 q_i^2}{2m} + \frac{\hbar^2 K_i^2}{8m} \right] \tag{10}$$

which concludes that $\Phi_n(N)$ is an eigenstate of $H_o(N)$ with E_n as its energy eigenvalue. While, at the first glance, Eqn. 10 indicates that E_n is purely a kinetic energy but the fact, that q values are constrained to satisfy $q \geq q_o = \pi/d$ (*cf.* Appendix-A, Eqn. A-13), indicates that HC interaction has an obvious control on E_n; of course this impact comes to surface only at low temperatures at which quantum effects start dominating the behavior of the system.

2.5 G-state configuration

In the light of our inference that $q \geq \pi/d$ (*cf.* Appendix-A, Eqn. A-13) and K can have any value between 0 and ∞, the G-state energy of a SIB can be obtained by using all $q_i = \pi/d_i$ and all $K_i = 0$ in Eqn. 10 and this renders

$$E_o = \sum_i^N \frac{h^2}{8md_i^2} = \sum_i^N \frac{h^2}{8m\text{v}_i^{2/3}} \qquad \text{with} \quad \sum_i^N \text{v}_i = \text{V} \quad \text{(constant)} \qquad (11)$$

where we use $d_i = \text{v}_i^{1/3}$ with v_i being the volume of the real space,- exclusively occupied by the $i-$th particle. In writing $\sum_i^N \text{v}_i = \text{V}$, we use the fact that each particle of the lowest possible q has largest possible quantum size $\lambda/2$ and, therefore, occupy largest possible v_i. While different particles can, in principle, occupy different volumes but the simple algebra reveals that E_o is minimum value for $\text{v}_1 = \text{v}_2 = ..\text{v}_N = \text{V}/N$ and we have

$$E_o = Nh^2/8md^2 = N\varepsilon_o. \qquad (12)$$

Since this means that all particles, identically, have $q = q_o$ and $K = 0$, use of these values in Eqn. 6 renders

$$\Phi_o(N) = \Pi_i^N \zeta_{q_o}(r_i) = \Pi_i^N \sin(q_o r_i) \qquad (13)$$

as the G-state wavefunction. It is clear from Eqns. 12 and 13 that each particle in the system represents a particle trapped in a box (cavity formed by its neighboring particle) of size d and it rests at the central point of this cavity. Since each $\sin(q_o r_i)$ in Eqn. 13 represents a kind of stationary matter wave (SMW) which joins with other SMWs of neighboring particles at the boundaries between the two cavities they occupy, the G-state wave function seems to be a macroscopically large size 3-D network of SMWs which modulates the *relative positions* of two particles in phase space and extends from one end of the container to another. In Section (5.0), we find that this network gets energetically stabilized due to some kind of collective binding and assumes different aspects of macroscopic wave function of S-state as envisaged by London [12]. Using this information with our results [Eqns. A-13 and A-14 of Appendix-A)] related to $< r >$ and $< \phi >$ (the expectation of relative position of two particles in $r-$ and $\phi-$spaces), we find that the G-state configuration of particles can be described by

$$< k >= 0 \qquad < r >= d \qquad \text{and} \qquad < \phi >= 2n\pi \quad \text{with} \quad n = 1, 2, 3... \qquad (14)$$

Evidently, particles in the G-state cease to have relative motion, collisional motion and inter-particle scattering. Since all of them have identically equal $< r >= d$ (nearest neighbor distance) with constant distance of 2π in the $\phi-$space, the G-state represents their close packed arrangement in real space and they are constrained to move only in order of their locations. In the light of these inferences and our conclusion that particles in this state have a kind of collective binding at $T \leq T_\lambda$ (*cf.* Section 5.0), it is evident

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that particles in the LT phase have mutual binding in all the three spaces ($r-$, $q-$ and $\phi-$).

2.6 Evolution of the system on cooling

In view of the condition, $\lambda/2 \leq d$ (*cf.* Appendix-A, Eqn. A-13), $(d - \lambda/2)$ for a SIB of fixed number density decreases with decreasing T and at certain $T = T_c$ it vanishes at large. Consequently, the system at T_c has $q = q_o$ for all particles and its state (Eqn. 6) is now expressed by

$$\Phi_n^S(N) = \Phi_o(N) \sum_P^{N!} [(\pm 1)^P \Pi_i^N \exp i(P\mathbf{K}_i.\mathbf{R}_i)] \tag{15}$$

For the reasons which become clear in Sections 5.0 and 6.0, the superscript S in $\Phi_n^S(N)$ refers to S-phase. Eqn. 15 implies that all the $N!$ micro-states of our system appearing in $\Phi_n(N)$ (Eqn. 6) merge into one and the entire system at T_c attains a kind of oneness as envisaged by Taubes [64]. This also reveals that: (i) the system at T_c has two separate components, say F1 and F2. While F1 (described by $\sum_P^{N!}$.. part of $\Phi_n^S(N)$) represents a gas of non-interacting quasi- particle excitations originating from the plane wave $K-$motions of particles, F2 (described by $\Phi_o(N)$) represents the system in its G-state (or $T = 0$ state) which has no motion except zero-point $q-$motions. The density of F1, obviously, decreases with cooling the system and reaches zero value at $T = 0$. In Section-7.0, this separation is used to explain the two fluid behavior of Phase-II.

2.7 Quantum Correlations

Quantum correlations, basically, originating from the wave nature of particles play an important role in relation to the behavior of a quantum system. These correlations can be expressed in terms what is known as *quantum correlation potential* (QCP) which can be obtained [65, 66] by comparing the partition function (under the quantum limits of the system), $Z_q = \sum_n \exp(-E_n/k_B T)|\Phi_n(S)|^2$ and its classical equivalent, $Z_c = \sum_n \exp(-E_n/k_B T) \exp(-U_n/k_B T)$. Here $\Phi_n(S)$ is given by Eqn. 15. The application of this method to our system is justified because our theory describes the system by symmetrized plane waves rendering $< V_{HC}(r) >= 0$ which indicates that the HC potential is completely screened out. Simplifying U_n, one easily finds that pairwise QCP has two components. The U_{ij}^s pertaining to k motion controls the $\phi = kr$ position of a particle and we have

$$U_{ij}^s = -k_B T_o \ln[2 \sin^2(\phi/2)] \tag{16}$$

where T has been replaced by T_o because T equivalent of k motion energy at all $T \leq T_\lambda$ is T_o.

U_{ij}^s has minimum value $(-k_B T_o \ln 2)$ at $\phi = (2n + 1)\pi$ and maximum value $(= \infty)$ at $\phi = 2n\pi$ occurring periodically at $\Delta\phi = 2n\pi$ (with $n = 1, 2, 3, ...$). Since U_{ij}^s always

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increases for any small change $\delta\phi$ in ϕ at its minimum value, as

$$\frac{1}{2}C(\delta\phi)^2 = \frac{1}{4}k_BT_o(\delta\phi)^2 \qquad \text{with force constant} \qquad C = \frac{1}{2}k_BT_o \qquad (17)$$

and particles experience a force $= -C\delta\phi$ which tries to maintain $\delta\phi = 0$ and the order of particles in ϕ-space is sustained. Since U_{ij}^s is not a real interaction like $V(r)$ which can manipulate d, the ϕ-space order is, therefore, achieved when cooling drives all q towards q_o.

The second component pertaining to $K-$motions can be expressed by [56 and 65]

$$U_{ij} = -k_BT\ln\left[1 + \exp\left(-2\pi|R' - R''|^2/\lambda_T'^2\right)\right] \qquad (18)$$

with $\lambda_T' = h/\sqrt{2\pi(4m)k_BT}$ being the thermal de Broglie wavelength pertaining to $K-$motions for which each particle appears to have $4m$ mass. Note that U_{ij} is identical to the quantum correlation potential for non-interacting particles [65]. It may be seen as the origin of the force that facilitates BEC of particles at the $K = 0$ state by driving them towards $K = 0$ point in $K-$space where U_{ij} has its minimum value $-k_BT\ln 2$.

3.0 Thermodynamic Behavior

3.1 Equation of state

In what follows from Eqn. 10, we can express the energy of a particle in our system as

$$\epsilon = \varepsilon(K) + \varepsilon(k) = \frac{\hbar^2K^2}{8m} + \frac{\hbar^2k^2}{8m} \qquad (19)$$

which can have any value between ε_o and ∞, Interestingly, this possibility exists even if $\hbar^2k^2/8m$ is replaced by the lowest energy ε_o of its $q-$motion since K can have any value between 0 and ∞ and we can use

$$\epsilon = \frac{\hbar^2K^2}{8m} + \varepsilon_o \qquad (20)$$

in the starting expressions of the standard theory of BEC [67, Ch. 7] to obtain

$$\frac{PV}{k_BT} = -\Sigma_{\varepsilon(K)}\ln\left[1 - z\exp\left(-\beta[\varepsilon(K) + \varepsilon_o]\right)\right] \qquad (21)$$

and

$$N = \Sigma_{\varepsilon(K)}\frac{1}{z^{-1}\exp\left(\beta[\varepsilon(K) + \varepsilon_o]\right) - 1} \qquad (22)$$

with $\beta = \frac{1}{k_BT}$ and fugacity

$$z = \exp\left(\beta\mu\right) \qquad (\mu = \text{chemical potential}). \qquad (23)$$

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Once again, by following the steps of the standard theory of BEC [67] and redefining the fugacity by

$$z' = z \exp(-\beta\varepsilon_o) = \exp[\beta(\mu - \varepsilon_o)] = \exp[\beta\mu'] \qquad \text{with} \quad \mu' = \mu - \varepsilon_o \qquad (24)$$

we easily have

$$\frac{P}{k_B T} = -\frac{2\pi(8mk_B T)^{3/2}}{h^3} \int_0^\infty x^{1/2} \ln(1 - z'e^{-x}) dx = \frac{1}{\lambda^3} g_{5/2}(z') \qquad (25)$$

and

$$\frac{N - N_o}{V} = \frac{2\pi(8mk_B T)^{3/2}}{h^3} \int_0^\infty \frac{x^{1/2} dx}{z'^{-1} e^x - 1} = \frac{1}{\lambda^3} g_{3/2}(z') \qquad (26)$$

where $x = \beta\varepsilon(K)$, $\lambda = h/(2\pi(4m)k_B T)^{1/2}$ and $g_n(z')$ has its usual expression. This reduces our problem to that of non- interacting bosons but with a difference. Firstly, we have m replaced by $4m$ and z by z'. Secondly, the theory of non-interacting bosons concludes $z = 1$ (or $\mu = 0$) for $T \leq T_\lambda$ and $z < 1$ (or $\mu < 0$) for $T > T_\lambda$, while our theory of interacting bosons fixes $z' = 1$ [or $\mu' = 0$ rendering $\mu = \varepsilon_o$ (Eqn. 24)] for $T \leq T_\lambda$ and $z' < 1$ (or $\mu' < 0$ demanding $\mu < \varepsilon_o$) for $T > T_\lambda$. In other words we have z' and μ' in place of z or μ used in the theory of non-interacting bosons [67]. As such we can use Eqns. 21 and 22 and Eqns. 25 and 26 to evaluate different thermodynamic properties. For example, using Eqns. 25 and 26, we find the internal energy $U = -\frac{\partial}{\partial\beta}(\frac{PV}{k_B T})|_{z,V}$ of the system. We have

$$U = \frac{3}{2} k_B T \frac{V}{\lambda^3} g_{5/2}(z') + N\varepsilon_o = U' + N\varepsilon_o \qquad (27)$$

where $U' = -\frac{\partial}{\partial\beta}(\frac{PV}{k_B T})|_{z',V}$, obviously, represents internal energy of $K-$motions, while $N\varepsilon_o$ comes from $k-$motions. Similarly, the Helmholtz free energy of the system can be expressed as

$$F = N\mu - PV = N\varepsilon_o + (N\mu' - PV) = N\varepsilon_o + F' \qquad (28)$$

with F' referring to that of non-interacting bosons. Following the standard methodology, we may now analyze F for the physical conditions for which it becomes critical and leads to superfluidity.

3.2 Onset of $K = 0$ condensate and T_λ

In what follows from Eqn. 28, F is the sum of: (i) $F' = F(K)$, representing the contribution of $K-$morions which define a system of non-interacting quantum quasi-particles of (*bosonic nature*) and $4m$ mass and (ii) $N\varepsilon_o = F(q)$ representing the zero-point energy of $q-$motions. Following the standard theory of noninteracting bosons [67], $F(K)$ is expected to become critical at

$$T_b = \frac{1}{4} T_{\text{BEC}} = \frac{h^2}{8\pi mk_B} \left(\frac{N}{2.61V}\right)^{\frac{2}{3}} \qquad (29)$$

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with the onset of BEC of particles in their $K = 0$ state. T_{BEC} in Eqn. 29 represents the usual point of BEC in a system of non-interacting bosons and $\frac{1}{4}$ factor signifies that each boson for its $K-$motion behaves like a particle of mass $4m$ and T_{BEC} varies as $\frac{1}{m}$. However, this onset would occur only when the particles at large have fallen into the G-state of the other component of their motion (*viz.* $q-$motions) because in an otherwise situation $K-$motions can gain energy from $q-$motion energy. In what follows from Section 2.6, a SIB assumes a state of $q = q_o$ for all particles at

$$T_o = \frac{h^2}{8\pi m k_B} \frac{1}{d^2} \tag{30}$$

which represents the T equivalent of the G-state energy ε_o. It is evident that as soon as all particles have $q = q_o$ state, further cooling of the system would immediately trigger the process of their fall in $K = 0$ state. Evidently, the effective temperature, at which particles have the state of $q = q_o$ and the onset of $K = 0$ condensate, should be

$$T_\lambda = \frac{h^2}{8\pi m^* k_B} \left[\frac{1}{d^2} + \left(\frac{N}{2.61\text{V}} \right)^{\frac{2}{3}} \right] \tag{31}$$

Here we replace m by m^* to account for the impact of physical parameters such as pressure (P) on T_λ. In this context it may be noted that, in spite of the fact that particles in our system to a good approximation represent HC particles moving freely on a surface of constant $-V_o$, one can not ignore the role of inter-particle interaction during their relative motion. We note that d, V and m^* are three quantities which may change with increasing P. While T_λ is expected to increase with increasing P for the usual decrease in the values of d and V, however, T_λ may show a reverse change if m^* increases with P. In this context we note that m^* for ^4He atoms in LHE-4 should increase with P for an obvious increase in the strength of inter-particle *attraction* with increasing P. Evidently, Eqn. 31 can explain the P dependence of T_λ of LHE-4.

3.3 Nature of Transition

When our system is cooled through T_λ, its particles move from their state of $q > \pi/d$ (*i.e.*, $\phi > 2\pi$) to that of $q = \pi/d$ (*i.e.*, $\phi = 2\pi$) (*cf.* Section-2.5). Evidently the system transforms from a state of random distribution of its particles in $\phi-$space to that of orderly distribution with $\phi = 2n\pi$ ($n = 1, 2, 3, ..$). This means that $\lambda-$transition represents an onset of order-disorder of particles in $\phi-$space accompanied by the BEC of particles in the state of $K = 0$ and $q = \pi/d$. Since BEC of particles in $K = 0$ state is not different from the BEC of non-interacting bosons (a well known second order transition [67]) and their order-disorder basically represents their self-organization in $\phi-$space which results from a reshuffle of their momenta, it is evident that $\lambda-$transition is a second order transition.

3.4 Free energy and order parameter of T_λ

In the light of Section 3.1, it is evident that free energy F of the system has two components and it can be expressed as

$$F = F(q) + F(K) \approx N\varepsilon_o + F(K) \tag{32}$$

with

$$F(K) = k_B T . \frac{2\pi(8mk_B T)^{3/2}}{h^3} \int_0^\infty x^{1/2} \ln\left(1 - z'e^{-x}\right)dx = k_B T \frac{1}{\lambda^3} g_{5/2}(z') \tag{33}$$

Eqn. 32 would also hold for $T > T_\lambda$, provided such a T is low enough to allow only insignificantly low number of particles in the excited states of momentum $q \geq 2q_o$. Guided by the experimental fact that a system like LHE-4 exhibit superfluidity and related properties below T_λ, F can be expressed as

$$F(T, \Omega) = F_o + \frac{1}{2}A\Omega^2 + \frac{1}{4}B\Omega^4 + \frac{1}{6}C\Omega^6 + \ldots\ldots \tag{34}$$

with Ω being the order parameter, F_o being the constant component of F and A, B, C, $etc.$ being the coefficients of its expansion which may depend on physical conditions such as T and P. However, it may be mentioned that it is $F(q)$ (not $F(K)$) which is basically responsible for the unique properties of a SIB below T_λ since a gas of non-interacting bosons is not expected to exhibit superfluidity and related properties in spite of its BEC into zero momentum state. Defining $A = \alpha.(T - T_\lambda)/T_\lambda$, we find that

$$\Omega(T) \propto \sqrt{\frac{T_\lambda - T}{T_\lambda}}. \tag{35}$$

However, since T_o represents the T equivalent of the G-state energy of a particle ε_o, there is hardly any excitation in the system at T_o. Consequently, the state of the system at T_o is effectively a $T = 0$ state and for this reason we can renormalize the temperature scale by replacing T in Eqn. 35 by $T^* = T - T_o$ and recast $\Omega(T)$ as

$$\Omega(T^*) \propto \sqrt{\frac{T_\lambda^* - T^*}{T_\lambda^*}} = \sqrt{\frac{T_\lambda - T}{T_\lambda - T_o}} \tag{36}$$

which rightly indicates that $\Omega(T^*)$ reaches its maximum value 1.0 at $T = T_o$.

Now we may identify the physical nature of Ω and to this effect our first choice is the fraction of the number of bosons $(n_{K=0}(T) = N_{K=0}(T)/N)$ condensed into the G-state defined by $q = q_o$ and $K = 0$. Applying the standard theory of BEC of non-interacting bosons [67] to F1, we find that $n_{K=0}(T)$ should vary as $[1 - (T/T_\lambda)^{3/2}]$. However, for our system $n_{K=0}(T)$ is replaced by $n_{K=0}(T^*)$ which varies as

$$n_{K=0}(T^*) = \frac{N_{K=0}(T^*)}{N} = \left[1 - \left(\frac{T^*}{T_\lambda^*}\right)^{3/2}\right] = \left[1 - \left(\frac{T - T_o}{T_\lambda - T_o}\right)^{3/2}\right] \tag{37}$$

Our next choices could be: (i) the strain in the interparticle bonds $[(d_T - d_\lambda)/d_\lambda]$ whose origin, relation with $(\mathbf{q}, -\mathbf{q})$ bound pair formation, volume expansion on cooling through T_λ, and energy gap has been discussed at length in Section-5.0, and (ii) $n^* = [N^*(T_\lambda) - N^*(T)]/N^*(T_\lambda)$ which represents the fraction of particles that move from their state of disorder in $\phi-$space to that of order on cooling though T_λ (*cf.* Eqn. 58 for an expression for $N^*(T)$). We analyze n^* and its relation with the energy gap in Section-5.0. One may find that all these quantities are inter-related with each other.

3.5 Single particle density matrix and ODLRO

Using Eqn. 15, we obtain the single particle density matrix,

$$\rho(\mathbf{R}^* - \mathbf{R}) = \frac{N}{V}\left[\frac{N_{K=0}(T^*)}{N} + \frac{V}{\lambda_T'^3}\exp\left[-2\pi.\frac{|\mathbf{R}^* - \mathbf{R}|^2}{\lambda_T'^2}\right]\right]\left(\frac{2}{V}\sin^2[\frac{\pi(r'' - r')}{d}]\right)$$

(38)

with $N_{K=0}(T^*)/N = n_{K=0}(T^*)$ (*cf.*, Eqn. 37). Here we use: (i) $\mathbf{q}_o.(r'' - r')^* = 2n\pi + \mathbf{q}_o.(r'' - r')$ and (ii) a renormalized T scale $T^* = T - T_o$.

While the term in big $(..)$ of Eqn. 38 represents the variation of density over a single antinodal region of the SMW of a particle, $N_{K=0}(T^*)$ stands for the number of particles condensed to the state of $K = 0$ and $q = q_o$; $\lambda_T' = h/\sqrt{2\pi(4m)k_BT}$ represents thermal wavelength attributed to K motions. We note that under the limit $|\mathbf{R}^* - \mathbf{R}| \to \infty$, the "one particle density matrix" $(\rho(\mathbf{R}^* - \mathbf{R}))$ has nonzero value $(N_{K=0}(T)/V)$ for $T < T_\lambda$ and zero value for $T \geq T_\lambda$ since $N_{K=0}(T^*)$ is $\neq 0$ for $T < T_\lambda$ and $= 0$ for $T \geq T_\lambda$. Evidently, our theory satisfies the criterion of Penrose and Onsager [35] for the occurrence of BEC in the G-state of the system defined by $K = 0$ and $q = \pi/d$ and agrees with the idea of ODLRO, spontaneous symmetry breaking and phase coherence advanced, respectively, by Yang [37], Goldstone [68] and Anderson [69].

3.6 Logarithmic singularity of specific heat

As shown in this section, this singularity is associated with the process which reshuffles particles from their random positions in $\phi-$space at T_λ^+ (just above T_λ) to their ordered locations at T_λ^- (just below T_λ). We assume that in this process about N_λ particles move from their $\phi = (2n\pi \pm \delta\phi_\lambda)$ positions at T_λ^+ to $\phi = (2n + 1)\pi$ at T_λ^-. This releases $\Delta\varepsilon$ energy responsible for the singularity and we have

$$\Delta\varepsilon = -N_\lambda k_B T_o\left[(\ln 2)\sin^2\left(\frac{2n\pi \pm \delta\phi_\lambda}{2}\right) - \ln 2\right]$$

(39)

Following the theories [70] of critical phenomenon we may define

$$\delta\phi_\lambda = \delta\phi_\lambda(o)|\zeta|^\nu\left[1 + a_2|\zeta|^2 + a_3|\zeta|^3\right]$$

(40)

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with $\zeta = \frac{T - T_\lambda}{T_\lambda}$. To a good approximation we have

$$\Delta\varepsilon = -N\left(\frac{T - T_\lambda}{T_\lambda}\right) k_B T_o \ln\left(\frac{\delta\phi_\lambda(o)|\zeta|^\nu}{2}\right)^2 \tag{41}$$

by using $\delta\phi_\lambda = \delta\phi_\lambda(o)|\zeta|^\nu$ and $N_\lambda = N\frac{T-T_\lambda}{T_\lambda}$; the latter expression is so chosen to ensure that $\Delta\varepsilon$ does not diverge at T_λ and it decreases with decreasing T through T_λ. Eqn. 41 gives

$$C_p(T \approx T_\lambda) \approx -\frac{N}{T_\lambda} k_B T_o [2\nu \ln|\zeta| + \ln(\delta\phi_\lambda(o)^2) - \ln 4 + 2\nu] \tag{42}$$

It may be mentioned that ν, $\delta\phi_\lambda(o)$, *etc.* may not have same values at $T > T_\lambda$ and at $T < T_\lambda$, obviously, because the relative configuration of the system on the two sides of T_λ differ significantly.

4.0 Thermal Excitations.

4.1 Feynman's approach

Defining an excited state of the system by $\psi = \sum_i f(r_i)\phi$ and the G-state by ϕ, Feynman [71] showed that the excited state energy is minimum for $f(r_i) = \exp(i\mathbf{k}.\mathbf{r}_i)$ and he obtained

$$E(Q)_{Feyn} = \frac{\hbar^2 Q^2}{2mS(Q)} \tag{43}$$

with $S(Q) =$ structure factor of the system. However, $E(Q)$ obtained from Eqn. 43 was found to be about two times the experimental value. Introducing back flow effect Feynman and Cohen [72] later found better agreement with experiments but with considerable discrepancy at higher Q. In this section we use Feynman's approach to our particles in $(\mathbf{q}, -\mathbf{q})$ configuration. We note that under the impact of an excitation a SMW pair in the G-state configuration of $(\mathbf{q}_o, -\mathbf{q}_o)$ and $K = 0$, expressed by $\zeta(q_o)$ (Eqn. A-17, with $k = 2q_o$ and $K = 0$) moves to a new configuration $(\mathbf{q}_o + \Delta\mathbf{q}, -\mathbf{q}_o + \Delta\mathbf{q})$ described by

$$\zeta_{q_o}^*(r) = \zeta(q_o)(r)\exp(i\mathbf{Q}.\mathbf{R})\exp(-i\varepsilon(Q)t/\hbar) \tag{44}$$

with $2\Delta\mathbf{q} = \mathbf{Q}$ which means that the impulse changes in the CM momentum and energy of both particles and its impact should be identified by $\mathbf{Q}$ not by $\Delta\mathbf{q}$. Eqn. 44 further shows that $f(R) = \exp(i\mathbf{Q}.\mathbf{R})$ is the real form of f which renders $\psi = \sum_i \exp(i\mathbf{Q}.\mathbf{R}_i)\phi$. Using these facts and recasting the relation, $-(\hbar^2/2m)\nabla^2 f(R) = \varepsilon \int g(R - R')f(R')d^3R'$ (Eqn.(11.25) of Feynman [71], p 329-30), for i-th particle we find $-(\hbar^2/8m)\nabla_i^2 f(R) = \varepsilon_i(Q)S(Q)f(R)$; note that our single particle energy operator is $-(\hbar^2/8m)\nabla_i^2$. This renders $(\hbar^2 Q^2/8m) = \varepsilon_i(Q)S(Q)$. Adding this relation to a similar relation for j-th particle of the pair gives

$$E(Q) = \varepsilon_i(Q) + \varepsilon_j(Q) = \frac{\hbar^2 Q^2}{4mS(Q)} = \frac{1}{2}E(Q)_{Feyn} \tag{45}$$

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which naturally explains $E(Q)_{expt}$ of $He - II$.

4.2 Aproach of mono-atomic chain

We note that a system like liquid ^4He is expected to exhibit: (i) no transverse mode because the shear forces between its particles are negligibly small, and (ii) only one branch of longitudinal mode because the system is isotropic. Further since the particles in the G-state of a SIB from a kind of close packed arrangement with $\Delta\phi = kr = 2n\pi$ (Section 2.5), we can visualize waves of ϕ-oscillations. Using a linear chain of atoms with nearest neighbor forces, we find that the frequency dispersion of ϕ-oscillations can be expressed by

$$\omega_\phi(Q) = \sqrt{(4\text{C})/\beta}|\sin(Qd/2)| \tag{46}$$

where Q is the wave vector and β is the measure of inertia for ϕ motion. Since $\phi-$positions of particles are restored by U_{ij}^s, C is given by Eqn. 17. We note that ϕ-oscillations can appear as the oscillations of r and q because $\delta\phi = 2q\delta r + 2\delta qr$; we have phonons when q remains unchanged at q_o, and omons (a new kind of quantum quasi-particle representing a phononlike wave of the **o**scillations of **m**omentum) coordinates when r remains fixed at d.

Evidently, $\omega_r(Q)$ of phonons can be represented, to a good approximation, by the dispersion of the elastic waves in a chain of identical atoms and it can be obtained from Eqn. 46 by replacing β and C by m and C^*, respectively. We have

$$C^* = 4\pi^2\text{C}/d^2 = 2\pi^2 k_B T_o/d^2 = \pi h^2/4md^4 \tag{47}$$

However, for a better accuracy d and C^* should, respectively, be considered descending and ascending functions of Q because increase in the energy of particles changed by an excitation reduces their WP size and decreases d and thereby increases C. Consequently, $E_{ph}(Q)$ can be expressed more accurately by

$$E_{ph}(Q) = \hbar\omega_r(Q) = \hbar\sqrt{4C(Q)/m}|\sin(Qd(Q)/2)| \tag{48}.$$

We note that use of $C^*(Q)$ and $d(Q)$ not only explains the experimentally observed $E_{ph}(Q)$ of $He - II$ but also accounts for its anomalous nature at low Q [56]. This aspect has been studied in detail in [73, 74]. However, since $d(Q)$ can not be smaller than σ, $d(Q)$ and $C(Q)$ are bound to become Q independent for $Q > \pi/\sigma$ and the maximum in $E_{ph}(Q)$ (*i.e.* the position of so called maxon) should fall at maxon $Q = Q_{max} = \pi/\sigma$ and $E_{ph}(Q)$ for $Q > \pi/\sigma$ and $< 2\pi/d$ should follow

$$E_{ph}(Q) = \hbar\omega_r(Q) = \hbar\sqrt{4C(Q_{max})/m}|\sin(Q\sigma/2)| \tag{49}$$

It is evident that phononlike dispersion is expected till the excitation wavelength Λ remain larger than d (*i.e.* $Q < 2\pi/d$). However, the momentum and energy of the excitation

would be carried by only a single particle, if $Q < 2\pi/d$ (*i.e.* $\Lambda < \sigma$) and this implies that $E(Q)$ for $Q > 2\pi/d$ would follow

$$E_{sp}(Q) = \hbar^2 Q^2/2m_F \tag{50}$$

which represents a kind of *single particle* dispersion with m_F being a kind of *effective mass that measures the effect of quantum correlation in* $(\mathbf{q}, \mathbf{-q})$ pair configuration. It is expected to be around $4m$ near $Q = 2\pi/d$ and then decrease slowly with increasing Q beyond this point.

The transition of $E(Q)$ from $E_{ph}(Q)$ to $E_{sp}(Q)$ would, obviously, occur at $2\pi/d < Q < 2\pi/\sigma$, before the $E_{ph}(Q)$ (Eqn. 49) meets its zero value at $2\pi/\sigma$. This implies that $E(Q)$ has to have its minimum (*identified as the roton minimum* for $He - II$) at a $Q = Q_{min}$ near the mid-point of $Q = 2\pi/d$ and $Q = 2\pi/\sigma$, *i.e.*

$$Q_{min} \approx \left[\frac{\pi}{d} + \frac{\pi}{\sigma} \right] \tag{51}$$

Evidently, Eqns. 48-51 represent a Landau type spectrum and use of Eqn. 47 in Eqn. 46 renders

$$v_p = v_g = \sqrt{\pi}h/2md \tag{52}$$

where v_p and v_g, respectively, represent the phase and group velocities of phonons at $Q \approx 0$.

We further find that the equation of motion of r_s (the r of s-th atom)

$$\partial_t^2 r_s = -\frac{1}{4}\omega_o^2[2r_s - r_{s-1} - r_{s+1}] \tag{53}$$

transforms into a similar equation for $p_s = \hbar q_s$ by operating $m\partial_t$ and this concludes

$$\omega_r(Q) = \omega_q(Q) \tag{54}$$

implying that the omon dispersion $\omega_q(Q)$ is not different from phonon dispersion $\omega_r(Q)$. Interestingly, as concluded in Section (5.0), an omon is an anti-phonon quantum quasi-particle.

5.0 Energy Gap, Self Energy and Bound Pairs

The system at $T \leq T_\lambda$ is separated into two components, F1 and F2 (*cf.* Section 2.6). While F1 depicts $K-$motions as quasi-particle excitations whose density decreases on cooling and vanishes at $T = 0$, F2 represents the G-state of the system and portrays zero-point $q-$motions with $q = q_o$. Evidently, the unique low T properties of our system that last even at $T = 0$, are related to F2 and to this effect we examine its evolution on cooling the system from T_λ to $T = 0$.

As concluded in Sections 2.5 and 2.6, particles in the system are locked at $\varepsilon_o(T_\lambda)$. However, when the system is cooled to $T \leq T_\lambda$, $\varepsilon_o = h^2/8md^2$ tends to have a value lower than $\varepsilon_o(T_\lambda)$ which, obviously, requires an increase d from d_λ to $d_T = d_\lambda + \Delta d$; here d_λ means d at $\lambda-$point. Naturally, this brings zero-point force ($f_o = -\partial_d \varepsilon_o = h^2/4md^3$) into operation which increases d by pushing two neighboring particles to an increased distance d_T against their ' inherent attractive force f_a. Consequently, the system has volume expansion on cooling around T_λ and this agrees exactly with experimentally observed negative thermal expansion co-efficient of liquid ^4He at $T \leq T_\lambda$ [1]. Evidently, the evolution of F2 on cooling is an inter-play of f_o and f_a. In the following we use this inference and other important aspects of F2 to present three inter-related pictures to conclude the formation of ($\mathbf{q}$, - $\mathbf{q}$) bound pairs, energy gap and self energy.

(1): The impact of f_o on the states of the system at $T \leq T_\lambda$, which is decided by its balance with f_a, can be understood by using $V(r_{ij})$ as a perturbation. We first demonstrate this for the states of a pair of particles and to this effect diagonalise (2x2) energy matrix defined by $E_{11} = E_{22} = \varepsilon_o$ and $E_{12} = E_{21} = \beta_o$ with $\beta_o = < V(r_{ij}) >$. We note that f_o is a consequence of $V_{HC}(r)$ clubbed with wave nature of particles which increases its operation from σ to $\lambda/2$, particularly, for the particles of $\lambda/2 > \sigma$ [59]. Naturally, it appears on the scene only at $T \leq T_\lambda$ where $\lambda/2 \to d_\lambda^+$ (little more than d decided by $V(r_{ij})$). The macro-orbitals of particles under these physical situations should, obviously, have their overlap to render non-zero $< V(r_{ij}) >$ and this is in line with our zero-order results $< V_{HC}(r) >= 0$. Evidently, one finds that the pair has two states of energy $\varepsilon_o \pm |\beta_o|$; in fact $|\beta_o|$ should better be replaced by $|\beta_o(T)|$ as the overlap of the macro-orbitals of two particles may depend on T. The states of energy $(\varepsilon_o - |\beta_o(T)|)$ and $(\varepsilon_o + |\beta_o(T)|)$ can, respectively, be identified as bonding (or paired) and antibonding (or unpaired) states. This follows the established approach of *Molecular Orbital Theory* [75] applied to a similar case in which two identical atomic orbitals form two molecular orbitals of bonding and anti-bonding nature. The pair is expected to be in bonding state provided the two particles remain locked in the relative configuration characterized by Eqn. 14.

Now we apply the same approach to the state of N particles by constructing a NxN matrix for $H_r(N)$ (representing the relative motion of N particles) with $[H_r(N)]_{mn} = \varepsilon_o$ for $m = n$, and $[H_r(N)]_{mn} = [V(r)]_{mn}$ for $m \neq n$ with $[V(r)]_{mn}$ having non-zero value only if m and n refer to two neighboring particles. Note that each particle has 6-12 (depending on the symmetry of their assumed spatial arrangement) nearest neighbors. The diagonalisation of this matrix renders N/2 energy levels with energy $> N\varepsilon_o(T_\lambda)$ (*the anti-bonding states*) and N/2 energy levels with energy $< N\varepsilon_o(T_\lambda)$ (*the bonding states*). It is natural that F2, separated from F1, assumes the lowest possible bonding state. Since the perturbative effect of $V(r)$ lowers ε_o of each particle, identically, all particle fall in the bonding state simultaneously and acquire a kind of collective binding.

(2). Although, the preceding analysis renders a good account for the origin of the collective binding of particles, however, a relation that helps in calculating its magnitude is

needed. To this effect, we note that the energy of F2 deceases from $N\varepsilon_o(T_\lambda)$ to $N.\varepsilon_o(T)$ when the system moves from its $\lambda-$point to $T < T_\lambda$. This concludes that energy of each particle falls by

$$\Delta\epsilon = [\varepsilon_o(T_\lambda) - \varepsilon_o(T)] \approx \frac{h^2}{4md_\lambda^2}\frac{d_T - d_\lambda}{d_\lambda} = 2\varepsilon_o\frac{d_T - d_\lambda}{d_\lambda} \qquad (55)$$

A simple analysis of the equilibrium between f_o and f_a reveals that half of the $\Delta\epsilon$ is stored in the system as strain energy in inter-particle bonds, while the remaining half is lost to surrounding. Since this happens to all particles, the net energy loss $(= N\Delta\epsilon/2)$ by F2 turns out to be their collective binding. Using the coherence property of the system (evident from $\Delta\phi = 2n\pi$), one may find that the effective binding per particle becomes much larger than K_BT [76], even when the real binding per particle is very small and this ensures the stability of our system with collective binding among its particles. Evidently, the entire system at $T < T_\lambda$ seems to represent a macroscopically large single molecule as envisaged by Foot and Steane [77] for the BEC state of trapped dilute gases. It is obvious that the stability of this state can not be disturbed by any perturbation of energy $< N\Delta\epsilon/2$ and we may define

$$E_g(T) = N\Delta\epsilon/2 = N\varepsilon_o\frac{d_T - d_\lambda}{d_\lambda} \qquad (56)$$

as an energy gap between the S- and N-states.

The strain energy $\Delta V_s = N\Delta\epsilon/2$, stored with the elongated bonds between nearest neighbors, obviously, represents the net increase in the potential energy $(= \Delta V_s)$ of particles and we call it as the self energy of the system. Since the strain $(\Delta d/d_\lambda = (d_T - d_\lambda)/d_\lambda$ arising due to increase in quantum size of each particle from d_λ to d_T) is an obvious function of the momentum q of each particle, we have $\Delta V_s = \Delta V_s(q_1, q_2, ...q_N)$ and this prepares the system to sustain phononlike waves of collective oscillations of momentum coordinates of particles (named as omons, *cf.* Section-4.2). Evidently, ΔV_s can also be recognized as the energy of omon field, The fact that ΔV_s increases with decreasing T implies that omon field intensity increases when phonon field intensity decreases and *vice versa* and this means that either the omon field intensity is the intensity of phonons condensed with the system or an omon is an *anti-phonon* quantum quasi-particle. We note that ΔV_s assumes it maximum value at $T = 0$ and serves as the source of collective motions at $T = 0$ when phonon cease to exist.

Since each particle in our system represents a $(\mathbf{q}, -\mathbf{q})$ pair, the above analysis concludes that all particles at $T \leq T_\lambda$ assume the states of $(\mathbf{q}, -\mathbf{q})$ bound pairs. Further in view of our results (*cf.*, Eqn. 14), which indicate that each pair of particles in F2 is locked at $< k >= 0$, $< r >= d$ and $< \phi >= 2n\pi$, our *bound pair* represents two particles bound in all the three ($k-$, $r-$ and ϕ) spaces. However, this does not imply that two particles in the system form a He_2 type diatomic molecule; the entire system assumes a state where

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particles interact with their neighbors identically by two body forces. We use the word bound pair because each particle is in a state of $(\mathbf{q},-\mathbf{q})$ pair.

(3). As inferred by Ulhenback and Gropper [66], a quantum system can be treated like a classical system by adding quantum correlation potential to the hamiltonian of the system. Using this inference and Eqn. 16, we find an alternative relation for $E_g(T)$. We note that F2, for its proximity with F1, has a small number of particles $N^*(T)$ in thermally excited $q-$motion states of $q \geq 2q_o$. These particles are, obviously, devoid of quantum correlations with particles in the G-state $(q = q_o)$. However, in the process of cooling, these particles fall to the G-state and establish quantum correlations with other particles of this state. Consequently, the energy of each particle falls below zero energy level by $-k_B T_o \ln 2$. Assuming that number of these particles decreases from $N^*(T_\lambda)$ to $N^*(T)$, we find

$$\Delta\epsilon(T) = -k_B T_o \ln 2 [N^*(T_\lambda) - N^*(T)] \tag{57}$$

as the net fall in energy due to quantum correlations. Here we have

$$N^*(T) = \frac{V}{4\pi^2} \left[\frac{2m}{\hbar^2} \right]^{3/2} \int\limits_{\varepsilon_c}^{\infty} \left[\exp\left(\frac{\varepsilon - \varepsilon_o}{k_B T} \right) - 1 \right]^{-1} \sqrt{\varepsilon} d\varepsilon \tag{58}$$

where $\varepsilon_c = \hbar^2 Q_c^2/2m$ (with $Q_c \approx 2\pi/\sigma$). Once again it is evident that half of the $\Delta\epsilon(T)$ energy released out is stored back in the system as strain energy in expanding nearest neighbor inter-particle bonds which indicates that the net fall in energy per particle is $\frac{1}{2}\Delta\epsilon(T)$. Since the negative value of $\frac{1}{2}\Delta\epsilon(T)$ means a kind of collective binding of particles and we have $E_g(T) = \frac{1}{2}\Delta\epsilon(T)$ is an alternative relation for $E_g(T)$. Its accuracy is corroborated by the fact that $\frac{1}{2}\Delta\epsilon(T)$ (Eqn. 57) and $E_g(T)$ (Eqn. 56) have closely equal value for $He-II$ at all $T < T_\lambda$ [56]. Note that Eqn. 58 is an approximate relation for $N^*(T)$, since it uses a free particle dispersion, $\varepsilon = \hbar^2 Q^2/2m$ which is valid only to a good approximation.

6.0 Energy Gap and its Consequences

In what follows from the Section-5.0, the energy of F2 at $T \leq T_\lambda$ can be expressed as $F(q) = N\varepsilon_o(T_\lambda) - E_g(T)$ where $N\varepsilon_o(T_\lambda)$ is constant. Evidently, the origin of different properties (including superfluidity and related aspects) of our system lies with $E_g(T)$ and this fact is used for the following analysis.

1. Superfluidity and Related Properties :

If two heads X and Y in the system have small T and P (pressure) differences, the equation of state is $E_g(X) = E_g(Y) + S\Delta T - V\Delta P$. Using $E_g(X) = E_g(Y)$ for equilibrium, we get

$$S\Delta T = V\Delta P \tag{59}$$

This reveals that : (i) the system should exhibit thermo-mechanical and mechano-caloric effects, and (ii) the measurement of η by capillary flow method performed under the condition $\Delta T = 0$ and of thermal conductivity (Θ) determined under $\Delta P = 0$ should reveal $\eta = 0$ and $\Theta \approx \infty$, respectively. As such the S-phase is expected to be a superfluid of infinitely high Θ.

Interestingly, several important aspects of our system can also be followed qualitatively from the configuration of F2. For example, we note that: (i) a close packed arrangement of particles in a fluid like system can have no vacant site, particularly, because two neighboring particles experience zero point repulsion which, naturally, means that the system should have very large Θ, (ii) the system can not have thermal convection currents for its large Θ and close packing of particles and this explains why $He - II$ does not boil like $He - I$, (iii) since particles in F2 can move only in order of their locations and cease to have relative motion, the system is bound to exhibit vanishingly small η, particularly, for their flow in narrow capillary, *etc.*) In the rotating fluid, however, particles moving on the neighboring concentric circular paths of quantized vortices have relative velocity as a source of natural viscous behavior. This explains both *viscosity and rotation paradoxes* [5]. As such the loss of viscosity in linear motion is not due to any loss of viscous forces among the particles, rather it is the property of the S-phase configuration in which particles cease to have relative or collision motion.

2. Critical Velocities and Stability of S-phase :

Using the same argument, which renders Eqn. 44, we find that the state function $\Phi_o(N)$ of the S-phase changes to $\Phi_o^*(N)$ when the system is made to flow with velocity $v_f = \hbar \Delta \mathbf{q}/m$. We have

$$\Phi_o^*(N) = \Phi_o(N) \exp\left(i\mathbf{K}.\Sigma_i^N \mathbf{R_i}\right) \exp\left[-i[N(\varepsilon_o + \varepsilon(K)) - E_g(T)]t/\hbar\right] \qquad (60)$$

with $2\Delta \mathbf{q} = \mathbf{K}$. This reveals that the S-state function remains stable against the flow unless its energy $Nmv_f^2/2 = N\varepsilon(K)$ overtakes the collective binding $E_g(T)$. We use this observation to explain critical velocity v_c for which the system loses superfluidity. Equating $E_g(T)$ and flow energy ($N.mv_f^2/2$ with $v_f = v_c$), we obtain the upper bound of v_c. We have

$$v_c(T) = \sqrt{[2E_g(T)/Nm]} \qquad (61)$$

A $v_c < v_c(T)$, at which the superfluid may show signs of viscous behavior, can be expected due to creation of quantized vortices. However, this cause would not destroy superfluidity unless energy of all vortices produced in the system exceeds $E_g(T)$.

3. Coherence Length :

Since the main factor responsible for the coherence of F2 is its configuration which locks the particles at $\Delta \phi = 2n\pi$ (*cf.* Eqn. 14) with collective binding $E_g(T)$, the *coherence*

length (not to be confused with healing length [5]), can be obtained from

$$\xi(T) = 1/mv_c(T) = h\sqrt{[N/2mE_g(T)]} \tag{62}.$$

4. Superfluid Density :

Correlating the *superfluid density*, ρ_s, as the order parameter of the transition, with $E_g(T)$ we find a new relation

$$\rho_s(T) = \frac{E_g(T)}{E_g(0)}\rho(T) = \frac{d_T - d_\lambda}{d_o - d_\lambda}\rho(T) \tag{63}$$

to determine $\rho_s(T)$ and *normal density*, $\rho_n(T) = \rho(T) - \rho_s(T)$. Evidently, $v_c(T)$, $\xi(T)$, and $\rho_s(T)$ can be obtained if we know $E_g(T)$ (Eqn. 56). Further since S-state function vanishes at the boundaries of the system, it is natural that $E_g(T)$ and $\rho_s(T)$ also vanish there.

5. Superfluid Velocity :

Concentrating only on the time independent part, Eqn. 60 can be arranged as

$$\Phi_o^*(N) = \Phi_o(N)\exp\left(iS(R)\right) \tag{64}$$

with

$$S(R) = \mathbf{K}.(\Sigma_i^N \mathbf{R}_i) \tag{65}$$

being the phase of the S-state. This renders

$$\mathbf{v}_s = \frac{\hbar}{2m}\nabla_{R_j}S(R) = \frac{\hbar\Delta\mathbf{q}}{m} \tag{66}$$

as a relation for the superfluid velocity; here we use the fact that $\nabla_{R_j}S(R)$ renders the momentum of the pair (not of a single particle). One may find that Eqn. 66 does not differ from the superfluid wave function presumed in the $\Psi-$theory of superfluidity [78] (*cf.* Section (2.3) of Reference [8]), of course for the well defined phenomenological reasons, $\Psi-$theory assumes $S(R)$ to be a complex quantity.

6. Quantized Vortices

Using the symmetry property of a state of bosonic system, Feynman [54, 71] showed that the circulation, κ, of the velocity field should be quantized, *i.e.*, $\kappa = \frac{nh}{m}$ with $n = 1, 2, 3, ...$ However, Wilks [1] has rightly pointed out that this account does not explain the fact that $He - I$ to which Feynman's argument applies equally well, does not exhibit quantized vortices. Using Eqn. 66, we find that

$$\kappa = \sum_i \mathbf{v}_s(i).\Delta\mathbf{r}_i = \frac{\hbar}{m}\sum_i \Delta\mathbf{q}_i.\Delta\mathbf{r}_i = \frac{nh}{m} \tag{67}$$

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by using the condition that $\sum_i \Delta\mathbf{q}_i.\Delta\mathbf{r}_i = 2n\pi$ which presumes that particles moving on a closed path maintain phase correlation. To this effect our theory reveals that particles have their ϕ-positions locked at $\Delta\phi = 2n\pi$ only in S-phase which indicates that only this phase can exhibit quantized vortices. However, since particles of N-phase have random distribution ($\Delta\phi \geq 2n\pi$) in ϕ-space, this phase can not sustain $\phi-$correlation and quantized vortices.

7. S-state and its Similarity with Lasers

We note that the system below T_λ defines a 3-D network of SMWs extending from its one end to another without any discontinuity. In lasers too these are the standing waves of electromagnetic field that modulate the probability of finding a photon at a chosen phase point. The basic difference between the two lies in the number of bosons in a single *anti-nodal region* (AR) of a SMW. In case of lasers this could be any number since photons are non-interacting particles but for a SIB like ^4He or ^{87}Rb one AR can have only one atom.

7. Consistency with Phenomenological Theories and Experiments

1. Two fluid theory

Since our system at $T \leq T_\lambda$ has two separated components F1 and F2, it can be identified to be a homogeneous mixture of two fluids. While F1, described by plane waves of momentum K, represents a gas of non-interacting quasi-particle excitations, F2 representing zero-point $q-$motions describes the G-state of the system where particles are locked with $< k > = 0$, $< r > = \lambda/2$ and $\Delta\phi = 2n\pi$ and consequently cease to have relative motion, or collisional motion essential for non-zero viscosity. Naturally, the thermodynamic properties, such as specific heat, non-zero entropy, *etc.*, as well as non-zero viscosity are contributed totally by F1. As such F1 has all properties of a N-fluid, while F2 having zero entropy, zero viscosity, *etc.* has the properties of S-fluid. Most interestingly our theory provides microscopic origin to the two fluid theory of Landau [14] and in spite of its separation into F1 and F2, each and every particle can be seen to participate in both fluids.

2. $\Psi-$ Theory

We find that superfluidity is basically a property of the G-state (F2 component) of our system represented by Eqn. 13 which can also be expressed as

$$\Phi_o(N) = \sqrt{n} \tag{68}$$

with $n = N/V$. However, when F2 is made to flow its state is given by $\Phi_o^*(N) = \Phi_o(N)\exp iS(R)$ (Eqn. 64). For the phenomenological reasons (such as the superfluid

density n_s is not always equal to n), we assume that the phase $S(R)$ (Eqn. 65) is a complex quatity given by

$$S(R) = \xi_r(R) + i\xi_i(R) \tag{69}$$

which renders $n_s = n \exp{(-2\xi_i(R))}$. Evidently, $\Phi_o^*(N)$ has the structure of $\Psi-$function that forms the basis of the well known $\Psi-$theory of superfluidity. This shows that our theory provides microscopic foundation to the highly successful $\Psi-$theory [78].

3. Properties of He-II

(i) *Thermodynamic properties :* An estimate of T_λ for LHE-4 by using Eqn. 31 (with $m^* = m$) renders a value falling around 2.1 K which agrees closely with experimental value 2.17 K. In view of an observation by Woods and Cowley [4], Feynman's relation $E(Q)_{Feyn} = \hbar^2 Q^2/2mS(Q)$ [71] renders $E(Q)$ values that are nearly two times of $E(Q)_{expt}$ (experimental $E(Q)$ for $He-II$). Evidently, our $E(Q)$ (Eqn. 45) equaling one half of the $E(Q)_{Feyn}$, ensures good matching with $E(Q)_{expt}$) for He-II and this implies that our theory can explain the thermodynamic properties of $He-II$ accurately. In addition, using our alternate set of relations for $E(Q)$ (Eqns.46-51), concluded by our theory, we find good agreement between theory and experiment for : (i) for the anomalous nature of phonon dispersion at low Q, velocity of sound at low Q [56 and 80].

The problem of explaining the experimentally observed logarithmic singularity in $C_p(T)$ of LHE-4 at T_λ has been a challenging task (*cf.* a remark by Feynman in his book [71, p.34]). Naturally, the fact that for the first time our theory succeeded in explaining it at quantitative level speeks for the accuracy of the microscopic foundation of our theory. Using the parameters of liquid ^4He in Eqn. 42 with $\nu = 0.55$ and $\delta\phi_\lambda(o) = \pi$, we find

$$C_p(\text{J/mole.K}) \approx -5.71 \ln|\zeta| - 10.35 = -A. \ln|\zeta| + B \tag{70}$$

which can be compared with experimental results $C_p = -5.355 \ln|\zeta| - 7.77$ for $T > T_\lambda$ and $C_p = -5.1 \ln|\zeta| + 15.52$ for $T < T_\lambda$ [79]. The fact that our A value agrees closely with experiments speaks of the accuracy of our theoretical result. With respect to our choice of $\nu = 0.55$ and $\delta\phi_\lambda(o) = \pi$, we note that: (i) $\delta\phi_\lambda$ originates basically from change in momentum $\Delta k \approx \xi^{-1}$ and ξ varies around T_λ as $|T - T_\lambda|^{-\nu}$; note that critical exponent ν lies in the range 0.55 to 0.7 [70]. For our choice of $\delta\phi_\lambda(o) = \pi$, we note that π is the largest possible value by which phase position of a particle changes. We note that experimental value of $B = -7.77$ (for $T > T_\lambda$) and $+ 15.52$ (for $T < T_\lambda$) can be obtained if $\delta\phi_\lambda(o)$ is chosen to be around 0.78π and 0.08π, respectively. These are consistent with the reasons of assymetry of this singularity as stated in Section-3.6 and the fact that $\phi-$fluctuations at $T > T_\lambda$ are expected to be much higher than those at $T < T_\lambda$.

(i) *Hydrodynamic properties :* Our theory provides microscopic foundations for: (i) two fluid theory of Landau (Section 7.1), (ii) the $\Psi-$theory of superfluidity (Section 7.2), (iii) the quantized vortices (Section 6.6), (iv) the vanishing of $\rho_s(T)$ at the boundaries of the

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system (Section 6.4), (v) the collective binding or energy gap (Section 5.0) which not only ensures the stability of S-state but also accounts for the critical velocities, coherence length, *etc.* (Section 6.0). Evidently, it can explain the hydrodynamic properties of superfluid SIB like $He - II$. While a brief analysis of the agreement between theory and experiments for $He - II$ is available in [56], a detailed quantitative study, being completed by our group [80], would be published soon.

As shown Fig.(1), we note that the T dependence of: (i) the fraction of particles condensed in $K = 0$ state, $n_{K=0}(T^*)$ [Curve A*, Eqn. 37] and (ii) $n^*(T) = [1 - N^*(T)/N^*(T_\lambda)]$ [Curve B, Eqn. 58] have an excellent agreement with similar nature of the order parameter ($\Omega(T^*)$ [Curve C*, Eqn. 36] of a second order transition. In addition they have similar agreement with the T variation of : (i) $\rho_s(T)$ (Curve E2) deduced from the experimentally observed second sound velocity of $He - II$ (taken from [5]), and (ii) the normalized inter-particle bond strain $d_T - d_\lambda / d_o - d_\lambda$ (curve E1) obtained from experimental values of $\rho(T)$ [1]. These facts, evidently, identify inter-particle bond strain as a basic aspect of the $\lambda-$transition, T^* as the effective temperature scale for the order parameter, and establish the accuracy of our theory in accounting for the unique properties of $He - II$. It is further corroborated by the fact that E1 and E2 curves do not agree with $n_{K=0}(T) = 1 - (T/T_\lambda)^{3/2}$ (Curve A) and $\Omega(T)$ [Curve C, Eqn. 35]. As such an overall agreement between experimental and theoretical results for LHE-4 establishes the accuracy of our theory.

8.0 Concluding Remarks

This paper presents an alternative approach to the microscopic understanding of a SIB. We find that each particle in the system has two motions, $q-$ and $K-$, because it represents a pair of particles moving with equal and opposite momenta ($\mathbf{q}$, -$\mathbf{q}$) at their CM which moves with momentum $\mathbf{K}$. Consequently, it should be described by a *macro-orbital*. The onset of $\lambda-$ transition is an order-disorder of particles in $\phi-$space followed simultaneously by their BEC in the state of $q = q_o$ and $K = 0$. It is a consequence of quantum correlations which drive $q-$values towards $q = q_o$ and $K-$values towards $K = 0$ and an inter-play of zero-point force f_o and inter-particle attraction f_a which leaves a kind of mechanical strain in the inter-particle bonds as the basic component of order parameter (Cf. Fig.1 curve E1). The system is expected to exhibit $-ve$ thermal expansion coefficient around T_λ as evinced by LHE-4.

The transition separates the two motions and the system behaves like a homogeneous mixture of two fluids: (i) F1, -representing a gas of quasi-particle excitations describing the plane wave $K-$motions of N particles and (ii) F2, -representing the G-state of the system where particles are locked at $< r >= d$, $\Delta\phi = 2n\pi$ and zero-point motions at $q = q_o$ and $K = 0$. The condensate fraction $n_{K=0}(T)$ rises smoothly from $n_{K=0}(T) = 0$ at $T = T_\lambda$, reaches $n_{K=0}(T) \approx 1.0$ around $T = T_o$ and $n_{K=0}(T) = 1.0$ at $T = 0$. The

HC interaction leading to excluded volume condition [51] pushes all particles to occupy identically equal volume and a state of identically equal $q = q_o$.

Superfluidity and related properties are, basically, the properties of $T = 0$ state of F2 component of a SIB which implies that $\lambda-$transition is a kind of quantum transition which occurs at non-zero T for the proximity of F2 with F1. Particles in F2 represent (**q**, -**q**) bound pairs, not only in $q-$space but also in $r-$ and $\phi-$spaces. They assume a kind of close packed arrangement with collective binding which leads the entire system to behave like a macroscopically large single molecule. The collective binding represents an energy gap between the S- and N-fluid phases. The S-state is consistent with microscopic uncertainty as evident from $q \geq \pi/d$ as well as macroscopic uncertainty since $\Phi_o(N)$ (Eqn. 13) vanishes at the boundaries of the system. The difference between the S- and N-fluid states can be identified with a difference in the ordered positions and motions of soldiers in an organized army platoon and a crowed. Our theory makes no assumption about the nature of BEC in a SIB. It reaches its all conclusions from the analysis of the solutions of its $N-$body Schrödinger equations.

In many respect, above summarized inferences of our theory differ from conventional theories which presume the existence of $p = 0$ condensate in the S-state of a SIB. Using two different theoretical approaches (**A1** and **A2** summarized in Section-1.0) and different possible mathematical tools, they conclude a maximum of $\approx 13\%$ $p = 0$ condensate in $He - II$ and $\approx 60\%$ in recently discovered BEC state of dilute trapped gases [81] and for the last several decades this conclusion has been regarded as basic origin of the unique properties of every superfluid SIB.

In summary, this paper provides an alternative theoretical framework concluding a different picture of S-phase of a SIB like LHE-4. Analyzing the wealth of experimental observations on the S-state of different SIB(s), one may agree that the wave nature of particles has wonderful capacity for their self organization in phase space at $\phi = 2n\pi$ with $< r > = \lambda/2 = d$ and our theoretical framework can reveal the truth of the low T behavior of widely different many body systems.

Finally, as discussed briefly in [57, 60], the framework of our theory unifies the physics of widely different many body systems of interacting bosons and fermions including low dimensional systems, atomic nucleus, newly discovered BEC states [81], *etc.* This is corroborated by our detailed studies of N HC particles in 1-D box [61] and microscopic theory of superconductivity [62] which not only accounts for the highest T_c that we know to-day but also reveals that the phenomenon may, in principle, occur at room temperature and co-exist with ferro-magnetism. For the first time our theory concludes [60, 62] that superfluid transition in liquid ^3He should occur around 2mK which matches almost exactly with experimental result. It may be emphasized that (**q**, -**q**) bound pairs of particles as concluded in [62] and the present paper should not be confused with Cooper pairs [82] since the two have several similarities and dissimilarities.

Appendix -A

Wave Mechanics of Two HC Particles and Macro-orbitals
detailed analysis on this topic avaible in [62]

The dynamics of two HC impenetrable particles P1 and P2 can be described by

$$\left(-\frac{\hbar^2}{2m}\sum_i^2 \nabla_i^2 + V_{HC}(r)\right)\psi(1,2) = E(2)\psi(1,2) \qquad (A-1)$$

As argued in Section 2.1, we may use $V_{HC}(r) \equiv A\delta(r)$ (Eqn. 3) and simplify the solution by using CM coordinates

$$\mathbf{r} = \mathbf{b}_2 - \mathbf{b}_1 \quad \text{and} \quad \mathbf{k} = \mathbf{p}_2 - \mathbf{p}_1 = 2\mathbf{q}, \qquad (A-2)$$

where $\mathbf{r}$ and $\mathbf{k}$, respectively, represent the relative position and relative momentum of P1 and P2, and

$$\mathbf{R} = (\mathbf{b}_2 + \mathbf{b}_1)/2 \quad \text{and} \quad \mathbf{K} = \mathbf{p}_2 + \mathbf{p}_1, \qquad (A-3)$$

where $\mathbf{R}$ and $\mathbf{K}$, similarly, refer to the position and momentum of their CM. Without loss of generality, Eqns. A-2 and A-3 also render

$$\mathbf{p}_1 = -\mathbf{q} + \frac{\mathbf{K}}{2} \quad \text{and} \quad \mathbf{p}_2 = \mathbf{q} + \frac{\mathbf{K}}{2}. \qquad (A-4)$$

We use these relations to express Eqn. A-1 as

$$\left(-\frac{\hbar^2}{4m}\nabla_R^2 - \frac{\hbar^2}{m}\nabla_r^2 + A\delta(r)\right)\Psi(r,R) = E(2)\Psi(r,R) \qquad (A-5)$$

with

$$\Psi(r,R) = \psi_k(r)\exp(i\mathbf{K}.\mathbf{R}). \qquad (A-6)$$

The HC interaction affects only $\psi_k(r)$ (the state of relative motion) which represents a solution of

$$\left(-\frac{\hbar^2}{m}\nabla_r^2 + A\delta(r)\right)\psi_k(r) = E_k\psi_k(r) \qquad (A-7)$$

with $E_k = E(2) - \hbar^2 K^2/4m$, while the CM motion $[\exp(i\mathbf{K}.\mathbf{R})]$ remains unaffected.

To find $\Psi(r,R)$ (Eqn. A-6), we treat $A\delta(r)$ as a step potential. Since $A\delta(r) = 0$ for $r \neq 0$, P1 and P2 can be represented, by independent plane waves but at $r = 0$ where $A\delta(r) = \infty$, we use the condition $\Psi(r,R)|_{r=0} = 0$. Following these points, a state of P1 and P2 can, in principle, be expressed by

$$\Psi(1,2)^{\pm} = \frac{1}{\sqrt{2}}[u_{\mathbf{p}_1}(\mathbf{r}_1)u_{\mathbf{p}_2}(\mathbf{r}_2) \pm u_{\mathbf{p}_2}(\mathbf{r}_1)u_{\mathbf{p}_1}(\mathbf{r}_2)]. \qquad (A-8)$$

However, $\Psi(1,2)^+$ (of $+ve$ symmetry for the exchange of two particles) has to be excluded as undesirable function because it does not satisfy $\Psi(r,R)|_{r=0} = 0$. But this leaves only $\Psi(1,2)^-$ of $-ve$ symmetry which does not fit with the bosonic character of our system. We encountered this problem in our recent study of the 1-D analogue of Eqn. A-7 in relation to the wave mechanics of two HC impenetrable particles in 1-D box [59]. Following this study we find that the state of P1 and P2 can be expressed by

$$\zeta(r,R)^\pm = \zeta_k(r)^\pm \exp\left(i\mathbf{K}.\mathbf{R}\right) \qquad (A-9)$$

with

$$\zeta_k(r)^- = \sqrt{2}\sin\left(\mathbf{k}.\mathbf{r}/2\right) \qquad (A-10)$$

of -ve symmetry, and

$$\zeta_k(r)^+ = \sqrt{2}\sin\left(|\mathbf{k}.\mathbf{r}|/2\right) \qquad (A-11)$$

of +ve symmetry.

$\zeta_k(r)^\pm$ represents a kind of *stationary matter wave* (SMW) which modulates the probability $|\zeta_k(r)^\pm|^2$ of finding two particles at their relative phase (ϕ) position $\phi = \mathbf{k}.\mathbf{r}$ in the $\phi-$space. Interestingly, the equality $|\zeta_k(r)^-|^2 = |\zeta_k(r)^+|^2$ renders an *important fact* that the relative configuration and relative dynamics of two HC particles is independent of their fermionic or bosonic nature and the requirement of a bosonic or fermionic symmetry should be enforced on the wave functions of their $\mathbf{K}-$motions or spin motions. We use this fact in constructing $N-$particle wavefunction in Section 2.3. In agreement with Eqn. A-4, the SMW character of $\zeta_k(r)^\pm$ reveals that: (i) two HC particles have equal and opposite momenta $(\mathbf{q},\text{-}\mathbf{q})$ in the frame attached to their CM which moves with momentum $\mathbf{K}$ in the laboratory frame, and (ii) their relative motion maintains a center of symmetry at their CM. This implies

$$\mathbf{r}_{CM}(1) = -\mathbf{r}_{CM}(2) = \frac{\mathbf{r}}{2} \quad \text{and} \quad \mathbf{k}_{CM}(1) = -\mathbf{k}_{CM}(2) = \mathbf{q} \qquad (A-12)$$

where $\mathbf{r}_{CM}(i)$ and $\mathbf{k}_{CM}(i)$, respectively, refer to the position and momentum of $i-$th particle with respect to the CM of two particles.

Since $\zeta(r,R)^\pm$ is an eigenstate of the momentum/energy operators of the relative and CM motions of P1 and P2 (*not of individual* P1/P2), it, evidently, represents a state of their *mutual superposition* (MS). Of course, one can have an alternative picture by presuming that each of P1 and P2 after their collision at $r = 0$ falls back on the pre-collision side and assumes a kind of *self superposition* (SS) state (*i.e.,* the superposition of pre- and post-collision states of one and the same particle). Since P1 and P2 on their collision exchange their momenta, SS state is also described by $\zeta(r,R)^\pm$ and this agrees with the fact that P1 and P2 are identical particles and we have no means to ascertain whether they exchanged their positions or bounced back after exchanging their momenta.

Evidently, $\zeta(r, R)^{\pm}$ can be used to identically describe the MS state of P1 and P2 or the SS states of P1 or P2 and this observation helps in developing the *macro-orbital* representation for a HC particle in a fluid.

Analyzing the 1-D motion of two HC particles, we recently, concluded [59] that the expectation value of the relative distance between two particles satisfies $< x > \geq \lambda/2$. Applying this inference to $\zeta_k(r)^{\pm}$ state of P1 and P2, we find that their $< r >$ should satisfy $< r > \geq \lambda/2$ for $\mathbf{k}||\mathbf{r}$ case and $< r > \geq \lambda/2 \cos\theta$ for general case because the relative dynamics of two particles, interacting through a central force, is equivalent to such motion in 1-D. Thus from the experimental view point two HC particles can reach a shortest distance, $< r >_o = \lambda/2$ and in this situation their individual locations (*cf.* Eqn. A-12) are given by $< \mathbf{r}_{CM}(1) >_o = - < \mathbf{r}_{CM}(2) >_o = \lambda/4$. Using similar result for their shortest possible distance in $\phi-$space and $< V_{HC}(r) >$, *etc.* we note that $\zeta_k(r)^{\pm}$ state is characterized by

$$< r > \quad \geq \quad \lambda/2 \qquad \text{or} \qquad q \quad \geq \quad q_o (= \pi/d), \qquad (A-13)$$

with d being the nearest neighbor distance of two particles,

$$< \phi > \quad \geq 2\pi, \qquad (A-14)$$

$$< V_{HC}(r) > \quad = \quad < A\delta(r) > \quad = 0, \qquad (A-15)$$

and

$$E(2) = \quad < H(2) > \quad = \left[\frac{\hbar^2 k^2}{4m} + \frac{\hbar^2 K^2}{4m} \right]. \qquad (A-16)$$

Although, as evident from Eqn. A-16, two HC particles in $\zeta(r, R)^{\pm}$ seems have only kinetic energy but this does not mean that HC interaction has no impact on $E(2)$. We note that HC interaction controls $E(2)$ through $q \geq q_o$ (Eqn. A-13) and this has been demonstrated in Section 2.5 and 2.6. Analyzing the other consequences of Eqn. A-13, we note the following :

(i) *Quantum size :* A HC particle of momentum q exclusively occupies $\lambda/2$ space if $\lambda/2 > \sigma$ because only then the two particles maintain $< r > \geq \lambda/2$. We call $\lambda/2$ as *quantum size* of a particle. It is evident that quantum size is not an absolute size of a particle, rather it depends on the relative momentum $k = 2q$ of two particles; as such it is the size of one particle of momentum q as seen by other particle of momentum $-q$ or *vice versa*.

(ii) *Zero-point force :* It is natural that two particles should experience a repulsive force if they try violate $< r > \geq \lambda/2$ and this force would come into operation only when our system has fixed $< r > = d$ and $\lambda/2$ is made to increase by some suitable process. In a many body system this happens when the system is cooled through a temperature below which $\lambda/2$ tend to cross d or its q tends to fall below its zero-point value $q_o = \pi/d$. We call this force a zero-point repulsion and it can be derived from zero-point energy of the particle $\varepsilon_o = h^2/8md^2$ corresponding $q = q_o$. Since this force operates only when the two

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particles tend have a distance shorter than $\lambda/2$ and it does not depend of σ, it appears that this force represents nothing but the HC repulsion whose operational range gets extended from σ to $\lambda/2$ due to wave nature of particles.

(iii) *Macro-orbital representation :* Since two HC particles in $\zeta(r,R)^{\pm}$ state either experience mutual repulsion (if they somehow have $<r><\lambda/2$), or no force (if $<r>\geq\lambda/2$), it is clear that each of them retains its independent identity. Evidently, each particle can be identified to be in its SS state which is represented by a kind of pair waveform $\xi \equiv \zeta^{\pm}(r,R)$ proposed to be known as *macro-orbital* described by

$$\xi_i = \sqrt{2}\sin[(\mathbf{q}_i.\mathbf{r}_i)]\exp(\mathbf{K}_i.\mathbf{R}_i), \qquad (A-17)$$

with i ($i=1$ or 2) referring to one of the two particles and r_i being equivalent to $r_{CM}(i)$ (Eqn. A-12) varying from $r_i=0$ to $r_i=\lambda/2$, while R_i being the CM position of $i-$th particle. It is evident that each particle in its *macro-orbital* representation has two motions, q and K. The plane wave $K-$motion can have any K ranging from 0 to ∞, while the q is constrained to satisfy Eqn. A-13 due to HC interaction. A *macro-orbital* identifies each particle as an entity of size $\lambda/2$ moving with momentum K. Since $\zeta(r,R)^{\pm}$ is an eigenfunction of the energy/momentum operator of the pair *not of individual particle*, each particle shares the pair energy $E(2)$ (Eqn. A-16) equally and we have

$$E_1 = E_2 = \frac{E(2)}{2} = \frac{\hbar^2 q^2}{2m} + \frac{\hbar^2 K^2}{8m} \qquad (A-18)$$

This evidently shows that each particle for its $K-$motion behaves as an entity of mass $4m$.

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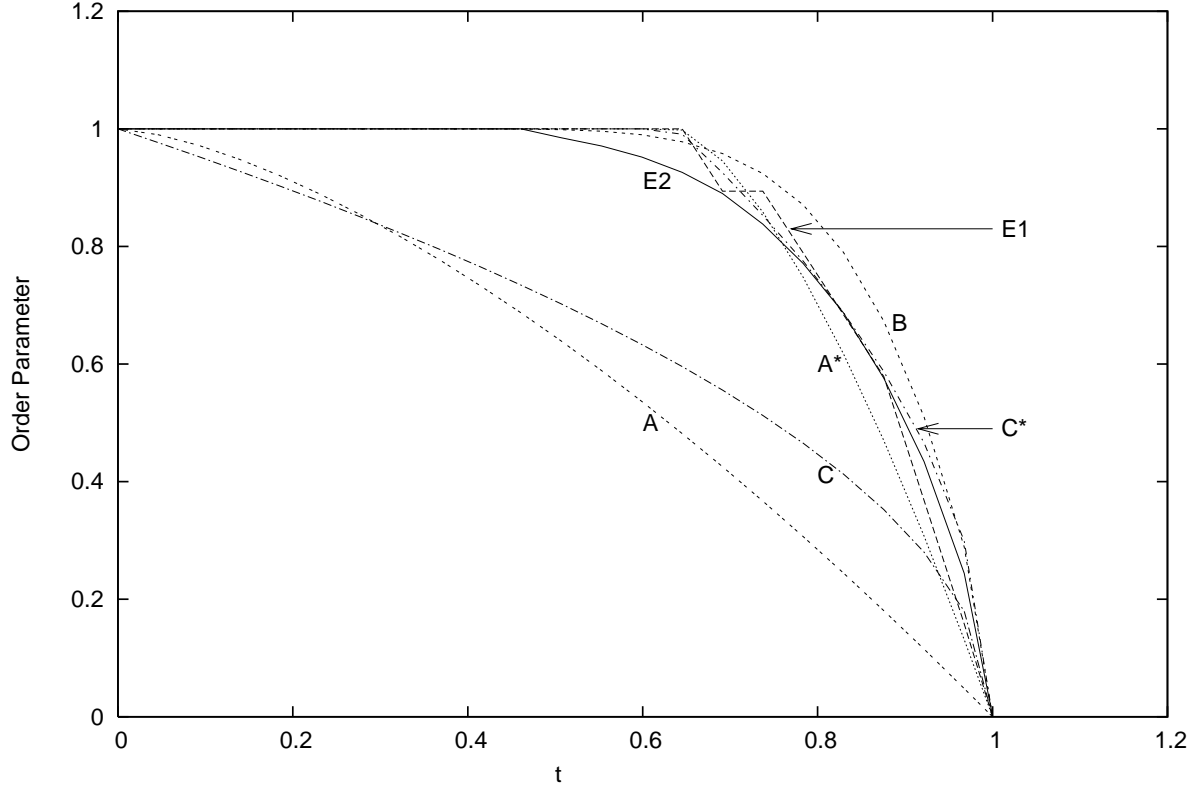**Figure -1** : $t = T/T_\lambda$ dependence of different representatives of the order parameter of $\lambda-$transition, *viz.*, $n_{K=0}(T^*)$ (Eqn. 37: *Curve*-A*), $n_{K=0}(T) = [1 - (T/T_\lambda)^{3/2}]$ (*Curve*-A), $n^*(T) = [1 \text{ -}N^*(T)/N^*(T_\lambda]$ (Eqn. 58: *Curve*-B), order parameter of second order phase transition $\Omega(T^*)$ (Eqn. 36: *Curve*-C*), $\Omega(T)$ (Eqn. 35: *Curve*-C), experimental $He - He$ bond strain $(d_T - d_\lambda)/(d_o - d_\lambda)$ (*Curve*-E1) and experimental $\rho_s(t)/\rho$ (*Curve*-E2).